δ- Best Approximation in 2-Normed Almost Linear Space

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Abstract:
The purpose of this paper is to introduce and discuss the concept of δ-best approximation in 2-normed almost linear space. A concept of δ-orthogonality in 2-normed almost linear space is also introduced and the relations between these two concepts are obtained in this paper.

Keywords:
2-normed almost linear space; best approximation in 2-normed almost linear space; δ-best approximation in 2-normed almost linear space; δ-orthogonalisation in 2-normed almost linear space.

1. Introduction

Diminnie, R. Freese [1] and many others developed new concept like linear 2-normed space. The notion of an almost linear space (als) was introduced by G. Godini [2]-[6]. All spaces involved in this work are over the real field \( \mathbb{R} \). S. Gahler, Y.J. Cho, C. S. Elumalai, R. Vijayaragavan [12]-[13] established some characterization of best approximation in terms of 2-semi inner products and normalized duality mapping associated with a linear 2-normed space. Basing on this we introduced a new concept called 2-normed almost linear space and established some results of best approximation in 2-normed almost linear space[17] and some results of best simultaneous approximation in 2-normed almost linear space [18]. Mehmet A¸çıkgöz [16] introduced the concept of approximation in generalized 2-normed linear space and established some results. Basing on this we introduced a new concept called δ- Best approximation in 2-normed almost linear space in this paper and we established some results on δ-best approximation and δ-orthogonalisation in 2-normed almost linear space.

2. Preliminaries
Definition: 2.1. Let \( X \) be an almost linear space of dimension \( \geq 1 \) and \( ||| \cdot |||: X \times X \rightarrow \mathbb{R} \) be a real valued function. If \( ||| \cdot ||| \) satisfy the following properties

i) \( ||| \alpha, \beta ||| = 0 \) if and only if \( \alpha \) and \( \beta \) are linearly dependent,

ii) \( ||| \alpha, \beta ||| = ||| \beta, \alpha ||| \),

iii) \( ||| a\alpha , \beta ||| = |a| \cdot ||| \alpha, \beta ||| \),

iv) \( ||| \alpha, \beta-\delta ||| \leq ||| \alpha, \beta-\gamma ||| + ||| \alpha, \gamma-\delta ||| \) for every \( \alpha, \beta, \gamma, \delta \in X \) and \( a \in \mathbb{R} \).
then \((X,|||\cdot|||)\) is called 2-normed almost linear space. ■

Example: 2.2. Let \(X = \mathbb{R}^n\). Let \(\alpha, \beta \in X\). Then \(\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)\) and \(\beta = (\beta_1, \beta_2, \beta_3, \ldots, \beta_n)\). Define 
\[|||\alpha, \beta||| = \sqrt{\sum_{i<j} (\alpha_i \beta_j - \beta_i \alpha_j)^2}\]
then \(|||\cdot|||\) satisfies all the properties of 2-normed almost linear space. Hence \((X, |||\cdot|||)\) is 2-normed almost linear space. ■

Example: 2.3. Let \(X = \mathbb{R}^n\). Let \(\alpha, \beta \in X\). Then \(\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)\) and \(\beta = (\beta_1, \beta_2, \beta_3, \ldots, \beta_n)\). Define 
\[|||\alpha, \beta||| = \sqrt{\sum_{i} (\alpha_i - \beta_i)^2}\]
then also \(|||\cdot|||\) satisfies all the properties of 2-normed almost linear space. Hence \((X, |||\cdot|||)\) is 2-normed almost linear space. ■

Example: 2.4. Let \((E,|| \cdot ||)\) be a normed linear space. Let \(X = \{\alpha \in E : \alpha \geq 0\}\). Then \(X\) is an als. Define 
\[|||\alpha, \beta||| = ||\alpha, \beta||\]
Where \(|| \cdot ||\) is 2-norm on \(E\). Then \(|||\cdot|||\) satisfies all the properties of 2-normed almost linear space. Hence \((X, |||\cdot|||)\) is 2-normed almost linear space. ■

Definition: 2.5. Let \(X\) be a 2-normed almost linear space over the real field \(\mathbb{R}\) and \(G\) a non empty subset of \(V_X\). For a bounded sub set \(A\) of \(X\) let us define

i) \(rad_G(A) = \inf_{g \in G} \sup_{a \in A} ||x, a - g||\) for every \(x \in X \setminus V_X\) and 2.1

ii) \(cent_G(A) = g_0 \in G : \sup_{a \in A} ||x, a-g_0|| = rad_G(A)\) for every \(x \in X \setminus V_X\). 2.2

The number \(rad_G(A)\) is called the chebyshev radius of \(A\) with respect to \(G\) and an element \(g_0 \in cent_G(A)\) is called a best simultaneous approximation or chebyshev centre of \(A\) with respect to \(G\). ■

Definition: 2.6. When \(A\) is a singleton say \(A = \{a\}, a \in X \setminus G\) then \(rad_G(A)\) is the distance of \(a\) to \(G\), denoted by \(dist(a,G)\) and defined by 
\[dist(a,G) = \inf_{g \in G} ||x, a-g||\] for every \(x \in X \setminus V_X\). 2.3

and \(cent_G(A)\) is the set of all best approximations of ‘\(a\)’ out of \(G\) denoted by \(P_G(a)\) and defined by 
\[P_G(a) = \{g_0 \in G : ||x, a-g_0|| = dist(a,G), for every x \in X \setminus V_X\}\] 2.4
Definition: 2.7. Let $X$ be a 2-normed almost linear space. The set $G$ is said to be proximinal if $P_G(a)$ is nonempty for each $a \in X \backslash V_X$.

It is well known that for any bounded subset $A$ of $X$ we have:

i) $\text{rad}_G(A) = \text{rad}_G(C_0(A)) = \text{rad}_G(\overline{A})$

ii) $\text{cent}_G(A) = \text{cent}_G(C_0(A)) = \text{cent}_G(\overline{A})$

Where $C_0(A)$ stands for the convex hull of $A$ and $\overline{A}$ stands for the closure of $A$.

Definition: 2.8. Let $X$ be a 2-normed almost linear spaces and $\phi \neq G \subset V_X$. We define $R_X(G) \subset X$ in the following way:

a $\in R_X(G)$ if for each $g \in G$ there exists $v_g \in V_X$ such that the following conditions are hold:

i) $\| x, a - v \| = \| x, v_g - g \|$ for each $v_g \in V_X$.

ii) $\| x, a - v \| \geq \| x, v_g - v \|$ for every $x \in X \backslash V_X$.

We have $V_X \subset R_X(G)$.

If $G_1 \subset G_2$ then $R_X(G_2) \subset R_X(G_1)$.

3. Main Results

Definition: 3.1 Let $X$ be a 2-normed almost linear space over the real field $\mathbb{R}$ and $G$ a nonempty subset of $V_X$ and $\delta > 0$. A point $g_0 \in G$ is said to be $\delta$-best approximation to $a \in A$ (a bounded subset of $X$) if $\| x, a - g \| \leq \| x, a - g_0 \| + \delta$ for all $g \in G$ and $x \in X \backslash V_X$.

Definition: 3.2 For a $\in A$, the set of all $\delta$-best approximation to a in $A$ denoted by $P_G(a, \delta)$ is defined as $P_G(a, \delta) = \{ g_0 \in G : \| x, a - g_0 \| \leq \| x, a - g \| + \delta, \text{for all } g \in G \text{ and } x \in X \backslash V_X \}$.

Theorem: 3.3 Let $G$ be a subspace of a 2-normed almost linear space $X$. Then the set $P_G(a, \delta)$ is bounded.

Proof: Let $g_1, g_2 \in P_G(a, \delta)$ and $a \in A$, then $\| x, a - g_1 \| \leq \| x, a - g \| + \delta$ and $\| x, a - g_2 \| \leq \| x, a - g \| + \delta$, for all $g \in G$ and $x \in X \backslash V_X$.

Now $\| x, g_1 - g_2 \| = \| x, g_1 - a + a - g_2 \| \leq \| x, a - g_1 \| + \| x, a - g_2 \|$

$\leq \| x, a - g \| + \delta + \| x, a - g \| + \delta$

$\leq 2 \| x, a - g \| + 2\delta$
\[ \leq 2 \, d(x, G) + 2\delta = M. \]

Therefore \[ \| x, g_1 - g_2 \| \leq M. \]

Hence \( P_G(a,\delta) \) is bounded. ■

Theorem: 3.4 Let \( G \) be a subspace of a 2-normed almost linear space \( X \) and \( a \in A \). Then the set \( P_G(a,\delta) \) is convex.

Proof: Let \( g_1, g_2 \in P_G(a,\delta) \) and \( 0 \leq \lambda \leq 1 \), then \[ \| x, a - g_1 \| \leq \| x, a - g \| + \delta \] and \[ \| x, a - g_2 \| \leq \| x, a - g \| + \delta \], for all \( g \in G \) and \( x \in X \setminus V_X \).

\[ \| x, a - \{ \lambda g_1 + (1 - \lambda) g_2 \} \| = \| x, a - \lambda g_1 - g_2 + \lambda g_2 \| \]
\[ = \| x, a - \lambda g_1 - g_2 + \lambda g_2 + \lambda a - \lambda a \| \]
\[ = \| x, \lambda (a - g_1) + (1 - \lambda) (a - g_2) \| \]
\[ \leq \lambda ( \| x, a - g \| + \delta ) + (1 - \lambda) ( \| x, a - g \| + \delta ) \]
\[ \leq \| x, a - g \| + \delta \]

This implies \( \lambda g_1 + (1 - \lambda) g_2 \in P_G(a,\delta) \).

Hence \( P_G(a,\delta) \) is convex. ■

Definition: 3.5 Let \( X \) be a 2-normed almost linear space, \( \delta > 0 \) and \( a, b \in A \). We call \( a \) is \( \delta \)-orthogonal to \( b \) and is denoted by \( a \perp_\delta b \) if and only if \[ \| x, a + \lambda b \| + \delta \geq \| x, a \| \] for all scalars \( |\lambda| \leq 1 \). ■

For any subsets \( G_1, G_2 \) of \( X \), \( G_1 \perp_\delta G_2 \) if and only if \( g_1 \perp_\delta g_2 \) for all \( g_1 \in G_1 \), \( g_2 \in G_2 \). ■

Theorem: 3.6 Let \( X \) be a 2-normed almost linear space and \( G \) be a subspace of \( X \) and \( \delta > 0 \). Then for all \( a \in A \), \( g_0 \in P_G(a,\delta) \) if and only if \( (a - g_0) \perp_\delta G \).

Proof: Suppose \( g_0 \in P_G(a,\delta) \).

Put \( g_1 = g_0 - \lambda g \) for \( g \in G \) and \( |\lambda| \leq 1 \).

Since \( g_0 \in P_G(a,\delta) \), and \( g_1 \in G \) we have \[ \| x, a - g_0 \| \leq \| x, a - g_1 \| + \delta \]
\[ \leq \| x, a - (g_0 - \lambda g) \| + \delta \]
\[ \leq \| x, a - g_0 + \lambda g \| + \delta \]
This implies \((a - g_0) \perp_\delta G\).

Conversely let \((a - g_0) \perp_\delta G\), then \(\|x, a - g_0 \| \leq \|x, a - g_0 + \lambda g_1 \| + \delta\) for all \(|\lambda| \leq 1\) and \(g_1 \in G\).

For any \(g \in G\) by putting \(g_1 = g_0 - g\) and \(\lambda = 1\) the last inequality implies
\[
\|x, a - g_0 \| \leq \|x, a - g || + \delta .
\]
This implies \(g_0 \in P_G(a, \delta)\).

**Definition:** 3.7 Let \(X\) be a 2-normed almost linear space and \(G\) be a subspace of \(X\) and \(\delta > 0\). Define \(\hat{G}_\delta = \{ a \in A : \|x, a \| \leq \|x, a - g || + \delta \text{ for every } g \in G \} = \{ a \in A : a \perp_\delta G \} \).

**Lemma:** 3.8 Let \(G\) be a subspace of a 2-normed almost linear space \(X\). Then for all \(a \in A\) and all \(\delta > 0\) we have \(g_0 \in P_G(a, \delta)\) if and only if \((a - g_0) \perp_\delta G\).

**Proof:** By theorem 3.6 we have \(g_0 \in P_G(a, \delta)\) if and only if \((a - g_0) \perp_\delta G\).

By the definition of \(\hat{G}_\delta\) we have \((a - g_0) \perp_\delta G\) if and only if \((a - g_0) \in \hat{G}_\delta\).

Now \((a - g_0) \perp_\delta G\) implies \(g_0 \in P_G(a, \delta)\) by theorem 3.6.

Therefore \(g_0 \in P_G(a, \delta)\) if and only if \((a - g_0) \in \hat{G}_\delta\).

**Theorem:** 3.9 Let \(G\) be a subspace of a 2-normed almost linear space \(X\), \(\delta > 0\) and \(\delta \geq \lambda\). Then \(\hat{G} \subseteq \hat{G}_\lambda \subseteq \hat{G}_\delta\) and \(\cap \hat{G}_\delta = \hat{G}\) for all \(\delta > 0\).

**Proof:** Let \(a \in \hat{G}\) then \(\|x, a \| \leq \|x, a - g || \) for all \(g \in G\) and \(x \in X \setminus V_x\).

Now \(\|x, a \| \leq \|x, a - g || \leq \|x, a - g || + \lambda, (\lambda > 0)\).

So we have \(a \in \hat{G}_\lambda\).

Hence \(\hat{G} \subseteq \hat{G}_\lambda\) \hspace{1cm} 3.1

Let \(a \in \hat{G}_\lambda\) then \(\|x, a \| \leq \|x, a - g || \leq \|x, a - g || + \lambda \leq \|x, a - g || + \delta, (\delta > 0)\).

This implies \(a \in \hat{G}_\delta\).

Therefore \(\hat{G}_\lambda \subseteq \hat{G}_\delta\) \hspace{1cm} 3.2

From 3.1 and 3.2 we have \(\hat{G} \subseteq \hat{G}_\lambda \subseteq \hat{G}_\delta\) . \hspace{1cm} 3.3

Now we prove that \(\cap \hat{G}_\delta = \hat{G}\) for all \(\delta > 0\).

From 3.3 we have \(\hat{G} \subseteq \cap \hat{G}_\delta\) for all \(\delta > 0\) \hspace{1cm} 3.4
Let $a \in \bigcap \hat{G}_\delta$ for all $\delta > 0$.

Then for all $\delta > 0, 0 \leq ||x, a|| \leq |||x, a - g||| + \delta$ for all $g \in G$ and $x \in X \setminus V_X$.

Now $0 \leq ||x, a|| \leq |||x, a - g||| + \frac{1}{n}$ for every $n \in \mathbb{N}$, for all $g \in G$ and $x \in X \setminus V_X$.

As $n \to \infty, |||x, a||| \leq |||x, a - g||| + \epsilon$ for all $g \in G$ and $x \in X \setminus V_X$.

Therefore $a \in \hat{G}$.

Hence $\bigcap \hat{G}_\delta \subseteq \hat{G}$, for all $\delta > 0$.

3.5

From 3.4 and 3.5 we have $\bigcap \hat{G}_\delta = \hat{G}$ for all $\delta > 0$. ■

Theorem: 3.10 Let $G$ be a subspace of a 2-normed almost linear space $X$. Then

i) If $\delta > 0, a \in A$ and $g \in G$ and $a \perp \delta g$ then $a \perp \epsilon g$ for all $\epsilon \geq \delta$ and

ii) If $a \perp \delta g$ and $| \lambda | < 1$ then $\lambda a \perp \delta \lambda g$.

Proof: i) Let $\delta > 0, a \in A$ and $g \in G$ and $a \perp \delta g$ then by definition 3.5 we have

$$|||x, a||| \leq |||x, a + \lambda g||| + \delta$$

when $| \lambda | \leq 1$ and $\delta > 0$.

Then $$|||x, a||| \leq |||x, a + \lambda g||| + \delta \leq |||x, a + \lambda g||| + \epsilon$$ (since $\epsilon \geq \delta$).

Therefore $a \perp \epsilon g$.

ii) Let $a \perp \delta g$ and $| \gamma | < 1$ then$|||x, a||| \leq |||x, a + \gamma g||| + \delta$

Multiplying both sides of 3.6 by $| \lambda |$ we get

$$| \lambda | |||x, a||| \leq | \lambda |||x, a + \gamma g||| + | \lambda | \delta \leq |||x, \lambda a + \lambda \gamma g||| + | \lambda | \delta$$

$$\leq |||\lambda x, \lambda a + \mu g||| + \delta$$

and so $|||\lambda x, \lambda a + \mu g||| \leq |||\lambda x, \lambda a + \mu g||| + \delta$.

Therefore $\lambda a \perp \delta \lambda g$. ■

Theorem: 3.11 Let $G$ be a subspace of a 2-normed almost linear space $X$.

If $\delta > 0, a \in A$ and $\epsilon \geq \delta$ then $P_G(a, \delta) \subseteq P_G(a, \epsilon)$.

Proof: Let $g_0 \in P_G(a, \delta)$ then by definition 3.1 we have

$$|||x, a - g_0||| \leq |||x, a - g||| + \delta$$

for all $g \in G$ and $x \in X \setminus V_X$ and $\delta > 0$.

Then $|||x, a - g_0||| \leq |||x, a - g||| + \delta$

$\leq |||x, a - g||| + \epsilon$ (since $\epsilon > \delta$)
Therefore \( g_0 \in P_G(a,\varepsilon) \).

Hence \( P_G(a,\delta) \subseteq P_G(a,\varepsilon) \). ■

**Reference**

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